

ON THE LIMIT SET OF ROOT SYSTEMS OF COXETER GROUPS AND KLEINIAN GROUPS

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ABSTRACT. The notion of limit roots of a Coxeter group W was recently introduced: they are the accumulation points of directions of roots of a root system for W . In the case where the root system lives in a Lorentzian space W admits a faithful representation as a discrete reflection group of isometries on a hyperbolic space; the accumulation set of any of its orbits is then classically called the limit set of W . In this article we show that the limit roots of a Coxeter group W acting on a Lorentzian space is equal to the limit set of W seen as a discrete reflection group of hyperbolic isometries. We aim for this article to be as self-contained as possible in order to be accessible to the community familiar with reflection groups and root systems and to the community familiar with discrete subgroups of isometries in hyperbolic geometry.

1. INTRODUCTION

Coxeter groups acting on a Euclidean vector space as a reflection group are precisely finite reflection groups [Bou68, Hum90]. In this case, the relation between reflection hyperplanes and the set of their normal vectors called *root system* is well understood and their interplay is the main tool to study those groups. Surprisingly, the duality roots/reflection hyperplanes is not very well exploited to study other cases of Coxeter groups. That is the case for instance for Coxeter groups seen as discrete groups generated by reflections on hyperbolic spaces. This article was motivated by the recent series of papers [HLR11, DHR13] in which some examples of the limit roots associated to a Coxeter group, acting as a discrete reflection groups on a Lorentzian space, looked like the limit set of a Kleinian group; see for instance the Apollonian circles in Figure 1 obtained as limit set of a root system.

On one hand, any Coxeter group W has a representation as a *discrete reflection subgroup* of the orthogonal group $O_B(V)$, where V is a finite dimensional vector space and B is a symmetric bilinear form. With such a representation of W arises a natural set of vectors Φ called a *root system*, which are unit B -normal vectors of the reflection hyperplanes associated to reflections in W . The root system Φ has an empty set of accumulation points, but the projective version $\hat{\Phi}$ of Φ , represented on an affine hyperplane \mathcal{H} , has an interesting set of accumulation points $E(\Phi)$ (see for instance Figure 1). The set $E(\Phi)$ is called the *set of limit roots of Φ* and its study was initiated in [HLR11] and continued in [DHR13]. Among other properties, it was

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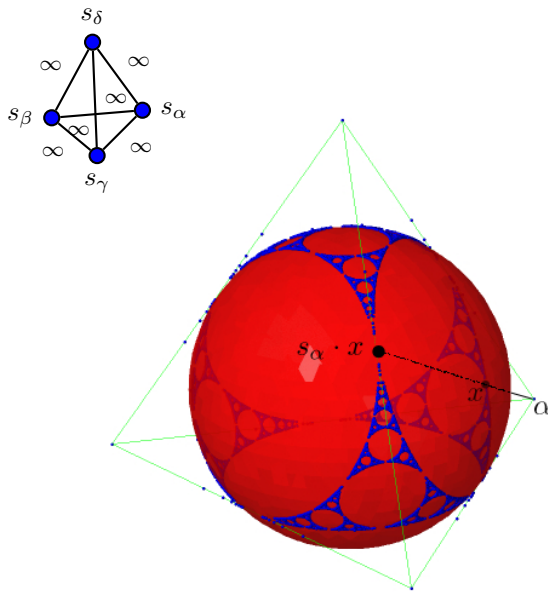


FIGURE 1. Picture of the first normalized roots of the root system Φ with diagram in the upper left of the picture and converging toward the set of limit roots $E(\Phi)$. The line $L(\alpha, s_\alpha \cdot x)$ represents the action of W on an element $x \in E(\Phi)$.

shown that $E(\Phi)$ lies on the isotropic cone $Q = \{v \in V \mid B(v, v) = 0\}$, that $E(\Phi)$ enjoys some fractal properties and the natural action of W on $E(\Phi)$ was studied.

On the other hand, when B is of signature $(n, 1)$, the couple (V, B) is called a *Lorentzian $(n + 1)$ -space* and Q is called the *light cone*, see for instance [Rat06, Chapter 3]. In this case, W is a discrete reflection subgroup of the group $\mathcal{I}(\mathbb{H}^n)$ of isometries of the hyperbolic n -space \mathbb{H}^n (see [Rat06]). There is an affine hyperplane \mathcal{H} of V such that $Q \cap \mathcal{H}$ is a ball and its interior $Q^- \cap \mathcal{H}$ is naturally identified with the *projective model* \mathbb{H}_p^n for \mathbb{H}^n (see details in §2). The *limit set of W* , denoted by $\Lambda(W)$, is the accumulation set of the W -orbit of a point $x \in \mathbb{H}_p^n$. The limit set $\Lambda(W)$ is of great interest in the theory of Kleinian groups and its generalizations, see for instance [Nic89, MT98, Rat06]. Our main result is the following.

Theorem 1.1. *If (V, B) is a Lorentzian space, the limit set $E(\Phi)$ of the root system Φ is equal to the limit set $\Lambda(W)$.*

The proof of this theorem basically follows from interpreting the results of [DHR13] into the language of hyperbolic geometry: a more precise statement, and its proof, are given in §3.4. It turns out that to do so brings out very interesting links that we feel should be further investigated. The last section §4 of this article is devoted to describe precisely the example in Figure 1 together with its relation with Apollonian gaskets.

We aim for this article to be accessible to the community familiar with reflection groups and root system and to the community familiar with discrete subgroups of isometries in hyperbolic geometry, so we will make a point to properly survey the objects and constructions mentioned above.

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2. LORENTZIAN AND HYPERBOLIC SPACES

The aim of this section is to survey the background we need on hyperbolic geometry. The presentation of the materials in this section is mostly based on [Rat06, Chapter 3 and §6.1], see also [BP92, Chapter A], [AVS93] and [Dav08, Chapter 6].

Let V be a real vector space of dimension $n + 1$ equipped with a symmetric bilinear form B . We will denote by $q(\cdot) = B(\cdot, \cdot)$ the quadratic form associated to B and by $Q := \{x \in V \mid q(x) = 0\}$ the *isotropic cone of B* , or equivalently, of q .

2.1. Lorentzian spaces. Suppose from now on that the signature of B is $(n, 1)$. The couple (V, B) is then called a *Lorentzian $(n + 1)$ -space* and Q is called the *light cone*. Moreover, the elements in the set $Q^- := \{v \in V \mid q(v) < 0\}$ are said to be *time-like*, while the elements in $Q^+ := \{v \in V \mid q(v) > 0\}$ are *space-like*¹; see the top picture in Figure 2 for an illustration.

A *Lorentzian transformation*² is a map on V that preserves B . So, in particular, a Lorentz transformation preserves Q , Q^+ and Q^- . It turns out that Lorentzian transformations are linear isomorphisms on V . We denote by $O_B(V)$ the set of Lorentzian transformations of V :

$$O_B(V) := \{f \in \text{GL}(V) \mid B(f(u), f(v)) = B(u, v), \forall u, v \in V\}.$$

The well-known *Cartan-Dieudonné Theorem* states that, since B is non-degenerate, an element of $O_B(V)$ is a product of at most $(n + 1)$ *B -reflections*: for a non-isotropic vector $\alpha \in V \setminus Q$, the *B -reflection associated to α* (or simply reflection since B is clear) is defined by the equation³

$$(1) \quad s_\alpha(v) = v - 2 \frac{B(\alpha, v)}{B(\alpha, \alpha)} \alpha, \quad \text{for any } v \in V.$$

We denote by $H_\alpha := \{v \in V \mid B(\alpha, v) = 0\}$ the orthogonal of the line $\mathbb{R}\alpha$ for the form B . Since $B(\alpha, \alpha) \neq 0$, we have $H_\alpha \oplus \mathbb{R}\alpha = V$. It is straightforward to check that s_α fixes H_α pointwise and that $s_\alpha(\alpha) = -\alpha$.

¹This vocabulary is borrowed from the theory of relativity, where $n = 4$.

²Lorentzian transformations are called *B -isometries* in [HLR11, DHR13], since in these articles B does not have necessarily $(n, 1)$ for signature.

³Observe that if B was a scalar product, this equation would be the usual formula for an Euclidean reflection.

2.2. Hyperbolic spaces. We fix a basis $\mathcal{B} = (e_1, \dots, e_{n+1})$ of V such that $q(v) = x_1^2 + x_2^2 + \dots + x_n^2 - x_{n+1}^2$, for any $v \in V$ with coordinates (x_1, \dots, x_{n+1}) in the basis \mathcal{B} . With this basis, V is often denoted by $\mathbb{R}^{n,1}$. The quadratic hyper-surface $\{v \in V \mid q(v) = -1\}$, called the *hyperboloid*, consists in time-like vectors and has two sheets. It is interesting to note that it is a differentiable surface (as the preimage of a regular value by the differentiable map $q : V \rightarrow \mathbb{R}$) and is naturally endowed with a Riemannian metric because $B(.,.)$ restricted to the tangent spaces of each sheet is definite positive. A *vector* v is *positive* if $x_{n+1} > 0$; the positive sheet,

$$\mathbb{H}^n := \{v \in V \mid q(v) = -1 \text{ and } x_{n+1} > 0\}$$

turns out to be a simply connected complete Riemannian manifold with constant sectional curvature equal to -1 (cf. [BP92, Theorem A.6.7]). This is *the hyperboloid model of the hyperbolic n -space*, see Figure 2. The distance function d on \mathbb{H}^n satisfies the equation $\cosh d(x, y) = -B(x, y)$.

2.2.1. Group of isometries. Observe that the group $O_B(V)$ acts on the quadratic hyper-surface $\{v \in V \mid q(v) = -1\}$. A Lorentz transformation is a *positive Lorentz transformation* if it maps time-like positive vectors to time-like positive vectors. So the group $O_B^+(V)$ of positive Lorentz transformations preserves \mathbb{H}^n and its distance, and the group of isometries $\mathcal{I}(\mathbb{H}^n)$ of \mathbb{H}^n is isomorphic to $O_B^+(V)$: any isometry of \mathbb{H}^n is the restriction to \mathbb{H}^n of a positive Lorentz transformation. Moreover, it is well known that $\mathcal{I}(\mathbb{H}^n)$ is generated by *hyperbolic reflections across hyperbolic hyperplanes*, which we now recall the definition.

2.2.2. Hyperbolic reflections. A linear subspace F of V is said to be *time-like* if $F \cap Q^- \neq \emptyset$, otherwise it is *space-like*. An *hyperbolic hyperplane* is the intersection of \mathbb{H}^n with a time-like hyperplane of V . Let H be a linear hyperplane in V and $\alpha \in V$ be a normal vector to H for the form B . Since $H \oplus \mathbb{R}\alpha = V$, we have necessarily that H is time-like if and only if $\alpha \in V$ is a space-like vector. A reflection $s_\alpha \in O_B(V)$ is an *hyperbolic reflection* if $\alpha^\perp = H$ is a time-like hyperplane or, equivalently, if α is a space-like vector of V . In this case, $s_\alpha \in O_B^+(V)$ and it restricts to an isometry of \mathbb{H}^n .

Remark 2.1. The fact that $s_\alpha \in O_B^+(V)$ for a space-like vector α follows from the fact that a reflection s_α is continuous and that it exchanges the two sheets (i.e. connected components) of the quadratic surface $\{v \in V \mid q(v) = -1\}$ if and only if H_α is space-like, i.e., if and only if α is a time-like vector.

2.3. The projective model. To make clear the link between hyperbolic geometry and the results of [HLR11, DHR13], we need to introduce another model for \mathbb{H}^n . Consider the unit open (Euclidean) n -ball embedded in the affine hyperplane $\mathbb{R}^n \times \{1\}$ of V :

$$D_1^n = \{v \in V \mid x_{n+1} = 1 \text{ and } x_1^2 + \dots + x_n^2 < 1\}$$

and the map p from D_1^n to \mathbb{H}^n , called the *radial projection*

$$p : D_1^n \rightarrow \mathbb{H}^n$$

where $p(v)$ is the intersection point of the line $\mathbb{R}v$ with \mathbb{H}^n (see Figure 2). A simple calculation shows that

$$p(v) = \frac{v}{\sqrt{|q(v)|}}.$$

The unit ball D_1^n endowed with the pullback metric with respect to p , i.e. which makes p an isometry, is a (non conformal) model \mathbb{H}_p^n for \mathbb{H}^n called the *projective ball model*⁴, see [Rat06, §6.1].

First, observe that using the equation for q in the basis \mathcal{B} , we have that $D_1^n \subseteq Q^-$. Let \mathcal{H} be the affine hyperplane directed by $\text{span}(e_1, \dots, e_n)$ and passing through the point e_{n+1} , then we get

$$D_1^n = Q^- \cap \mathcal{H},$$

with boundary $Q \cap \mathcal{H}$. The next proposition follows from the previous discussion and [Rat06, Equation (6.1.2)].

Proposition 2.2. *The projective model \mathbb{H}_p^n has underlying space $D_1^n = Q^- \cap \mathcal{H}$ and its boundary $\partial\mathbb{H}_p^n$ is $Q \cap \mathcal{H}$. Moreover, $p : \mathbb{H}_p^n \rightarrow \mathbb{H}^n$ is an isometry whose inverse is*

$$p^{-1}(v) = \frac{v}{x_{n+1}} = (x_1/x_{n+1}, \dots, x_n/x_{n+1}, 1).$$

This proposition is illustrated for $n + 1 = 2$ and $n + 1 = 3$ in Figure 2.

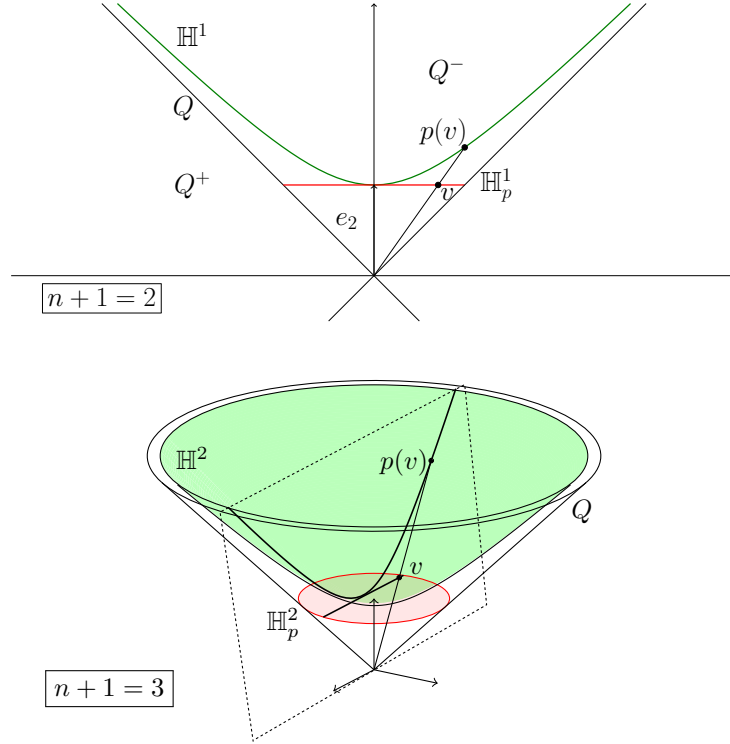


FIGURE 2. Pictures of Lorentzian spaces of dimension $n + 1 = 2$ and $n + 1 = 3$ with the hyperbolic spaces \mathbb{H}^n , the projective model \mathbb{H}_p^n , the radial projection p and a time-like plane H cutting \mathbb{H}^2 and \mathbb{H}_p^2 .

⁴This model is also sometimes called the *Beltrami-Klein model* in the literature.

2.3.1. Hyperplanes, reflections and isometries. The projective model gives us this easy description of hyperplanes: an *hyperbolic hyperplane* in \mathbb{H}_p^n is simply the intersection of a time-like linear hyperplane of V with \mathbb{H}_p^n . Let $\mathcal{I}(\mathbb{H}_p^n)$ be the group of isometries of \mathbb{H}_p^n .

Corollary 2.3. *The conjugation by p is an isomorphism from $\mathcal{I}(\mathbb{H}^n)$ to $\mathcal{I}(\mathbb{H}_p^n)$: for $\varphi \in \mathcal{I}(\mathbb{H}^n)$ and a point $v \in \mathbb{H}_p^n$, $\varphi \cdot v := p^{-1} \circ \varphi \circ p(v)$ defines the isometric action of φ on \mathbb{H}_p^n . Moreover $\varphi \cdot v$ is the intersection point of the linear line $\mathbb{R}\varphi(v)$ with the ball D_1^n .*

In particular, if $\alpha \in V$ is a space-like vector, then $s_\alpha \cdot v = p^{-1} \circ s_\alpha \circ p(v)$ is the hyperbolic reflection in $\mathcal{I}(\mathbb{H}_p^n)$ of v across the time-like hyperplane α^\perp .

Proof. This result follows immediately from Proposition 2.2. Let us detail the “moreover part”. Let (x_1, \dots, x_{n+1}) be the coordinates of v , and (y_1, \dots, y_{n+1}) the coordinates of $\varphi(v)$, in the basis \mathcal{B} . Since $v \in D_1^n$ and $\varphi \in \mathcal{O}_B^+(V)$, we have that $x_{n+1} = 1$ and $y_{n+1} > 0$. Therefore, $\varphi(v) \in Q^-$ and $\varphi(v)/y_{n+1} \in Q^- \cap \mathcal{H} = D_1^n$ is the intersection point of $\mathbb{R}\varphi(v)$ with the ball D_1^n . Now remember the formula for p^{-1} in Proposition 2.2: we have

$$\begin{aligned} \varphi \cdot v &= p^{-1} \circ \varphi \circ p(v) = p^{-1} \circ \varphi \left(\frac{v}{\sqrt{|q(v)|}} \right) \\ &= p^{-1} \left(\frac{1}{\sqrt{|q(v)|}} \varphi(v) \right) = p^{-1}(\varphi(v)) = \frac{\varphi(v)}{y_{n+1}}. \end{aligned}$$

□

2.4. Limit sets of discrete groups of hyperbolic isometries. The notion of *limit set* is central to study the dynamics of discrete groups of hyperbolic isometries: it provides an interesting topological space on which the group naturally acts, and the properties of the limit set characterize some properties of the group. We recall below the definition and some basic features of limit sets; see for example [Rat06, §12.1], [VS93, §2] or [Nic89, §1.4] for details.

Given a hyperbolic isometry φ of \mathbb{H}_p^n (with underlying space D_1^n), we extend the action of φ to the closed ball $\overline{D_1^n}$. Note that φ preserves the boundary $\partial\mathbb{H}_p^n = Q \cap \mathcal{H}$ of \mathbb{H}_p^n .

Let $\Gamma \subseteq \mathcal{I}(\mathbb{H}_p^n)$ be a discrete group of hyperbolic isometries. For x a point in the closed ball $\overline{D_1^n}$, the following are equivalent:

- x is an accumulation point of the orbit $\Gamma \cdot x_0$ for some $x_0 \in \mathbb{H}_p^n$;
- x is an accumulation point of the orbit $\Gamma \cdot x_0$ for any $x_0 \in \mathbb{H}_p^n$;
- x is in $\partial\mathbb{H}_p^n \cap \overline{\Gamma \cdot x_0}$ for some $x_0 \in \mathbb{H}_p^n$.

Such a point is called a *limit point* of Γ .

Definition 2.4. The set $\Lambda(\Gamma)$ of limit points is called *the limit set* of Γ .

The fact that an orbit has no accumulation points inside the open ball D_1^n is clear, since the group is discrete. The fact that the limit set of an orbit does not depend on the chosen point follows from the relation between hyperbolic and Euclidean distances, see [Rat06, Theorem 12.1.2].

The limit set $\Lambda(\Gamma)$ is clearly closed and Γ -stable. Many general properties are known for limit sets of discrete groups of hyperbolic isometries. For example, either

$\Lambda(\Gamma)$ is finite, in which case $|\Lambda(\Gamma)| \leq 2$ and Γ has a finite orbit in the closed ball $\overline{D_1^n}$, see [Rat06, Theorem 12.2.1], or $\Lambda(\Gamma)$ is the smallest non-empty Γ -invariant closed subset of $\partial\mathbb{H}_p^n$. In [DHR13], the authors show an analog property for the set of limit roots of a root system, which we will define below.

3. COXETER GROUPS AND HYPERBOLIC GEOMETRY

The aim of this section is to present constructions and results related to root systems, and that are taken out from [Dye12, HLR11, DHR13], into the language of hyperbolic geometry. This naturally leads to a proof of Theorem 1.1.

Recall that a Coxeter system (W, S) is such that $S \subseteq W$ is a set of generators for the Coxeter group W , subject only to relations of the form $(st)^{m_{s,t}} = 1$, where $m_{s,t} \in \mathbb{N}^* \cup \{\infty\}$ is attached to each pair of generators $s, t \in S$, with $m_{s,s} = 1$ and $m_{s,t} \geq 2$ for $s \neq t$. We write $m_{s,t} = \infty$ if the product st has infinite order. In the following, we always suppose that the set of generators S is finite. If all the $m_{s,t} = \infty$ then we say that W is a *universal Coxeter group*.

It turns out that any Coxeter group can be represented as a *discrete reflection subgroup* of $O_B(V)$ for a certain pair (V, B) : Coxeter groups are the discrete reflection groups associated to *based root systems* [Vin71, Theorem 2] (see [Kra09, §1 and Theorem 1.2.2] for a recent exposition of this result).

From now on, we fix (V, B) to be a Lorentzian $(n+1)$ -space. Note that most of the results from [Dye12, HLR11, DHR13] we recall in the two former sections are valid for arbitrary (V, B) .

3.1. Based root systems and geometric representations. Let us cover the basis on based root systems (see for instance [HLR11, §1] for more details). A *simple system* Δ is a finite subset of V such that:

- (i) Δ is positively independent: if $\sum_{\alpha \in \Delta} \lambda_\alpha \alpha = 0$ with all $\lambda_\alpha \geq 0$, then all $\lambda_\alpha = 0$;
- (ii) for all $\alpha, \beta \in \Delta$, with $\alpha \neq \beta$, $B(\alpha, \beta) \in]-\infty, -1] \cup \{-\cos(\frac{\pi}{k}), k \in \mathbb{Z}_{\geq 2}\}$;
- (iii) for all $\alpha \in \Delta$, $B(\alpha, \alpha) = 1$.

Denote by $S := \{s_\alpha \mid \alpha \in \Delta\}$ the set of B -reflections associated to elements in Δ . Let W be the subgroup of $O_B(V)$ generated by S , and $\Phi = W(\Delta)$ be the orbit of Δ under the action of W .

The pair (Φ, Δ) is called a *based root system in (V, B)* , for simplification we will often use the term *root system* instead of based root system; $\Phi^+ = \text{cone}(\Delta) \cap \Phi$ is its set of *positive roots*; its *rank* is the cardinality of Δ , i.e., the cardinality of S . Vinberg [Vin71, Theorem 2] shows that (W, S) is always a Coxeter system and W is a discrete reflection group in $O_B(V)$. Such a representation of a Coxeter group is called a *geometric representation*. Conversely, it is well known that any (finitely generated) Coxeter group can be geometrically represented with a root system, see for instance [Hum90, Chapter 5].

3.2. Representations as discrete reflection groups of hyperbolic isometries. A geometric representation of a Coxeter group W as a discrete subgroup of $O_B(V)$ in a Lorentzian $(n+1)$ -space yields a faithful representation of W as a discrete subgroup of isometries of \mathbb{H}^n that is generated by hyperbolic reflections;

and therefore, by conjugation by the radial projection (Corollary 2.3), it also provides a faithful representation of W as a discrete subgroup of $\mathcal{I}(\mathbb{H}_p^n)$ generated by reflections.

The key is to observe that $W \subseteq O_B(V)$ is in fact a subgroup of $O_B^+(V)$, the group of positive Lorentz transformations: from §2.2.1, we know that $O_B^+(V)$ is isomorphic to $\mathcal{I}(\mathbb{H}^n)$ by restriction to \mathbb{H}^n .

Proposition 3.1. *Let (Φ, Δ) be a root system in the Lorentzian $(n+1)$ -space (V, B) with associated Coxeter system (W, S) . Then $W \subseteq O_B^+(V)$ and this geometric action of W on the $(n+1)$ -Lorentzian space preserves \mathbb{H}^n . This yields a restricted representation of W on $\mathcal{I}(\mathbb{H}^n)$ that is faithful and discrete. Consequently, the projective action⁵ of W on \mathbb{H}_p^n is also faithful and discrete.*

Moreover, the action of W on \mathbb{H}^n (resp. \mathbb{H}_p^n) is generated by reflections across the hyperbolic hyperplanes $\alpha^\perp \cap \mathbb{H}^n$ (resp. $\alpha^\perp \cap \mathbb{H}_p^n$) for all $\alpha \in \Delta$.

Proof. Since Δ is constituted of space-like vectors, the hyperplanes α^\perp are time-like. From §2.2.2 we know therefore that $s_\alpha \in O_B^+(V)$ for all $\alpha \in \Delta$. Since W is generated by $S = \{s_\alpha \mid \alpha \in \Delta\}$, we have necessarily that $W \subseteq O_B^+(V)$. \square

Remark 3.2. The relative position between two hyperbolic hyperplanes has a nice characterization using their “normal vectors”. Let α and β be two space-like linearly independent vectors such that $B(\alpha, \alpha) = B(\beta, \beta) = 1$. So we know that α^\perp and β^\perp are time-like hyperplanes; moreover, denote by $H_\alpha = \alpha^\perp \cap \mathbb{H}^n$ and $H_\beta = \beta^\perp \cap \mathbb{H}^n$, then:

- (i) H_α and H_β intersect if and only if $B(\alpha, \beta) \in]-1, 1[$, and in which case their dihedral angle is $\arccos |B(\alpha, \beta)|$,
- (ii) H_α and H_β are parallel if and only if $B(\alpha, \beta) = \pm 1$,
- (iii) H_α and H_β are ultra-parallel if and only if $|B(\alpha, \beta)| > 1$ and in such case their distance in \mathbb{H}^n is $\cosh^{-1}(|B(\alpha, \beta)|)$.

(The statement (i) follows from Theorem 3.2.6 (see also the discussion that follows in §3.2 and §6.4) of [Rat06]; the statement (ii) follows from Theorem 3.2.9 of [Rat06] and the statement (iii) from Theorems 3.2.7 and 3.2.8 of [Rat06].)

3.3. Limits of roots. From now on, we fix a root system (Φ, Δ) in the Lorentzian $(n+1)$ -space (V, B) with associated Coxeter system (W, S) . We assume⁶ that $\text{span}(\Delta) = V$; such a root system is infinite and called a *weakly hyperbolic root system*.

The norm of any injective sequence of roots goes to infinity, so Φ does not have accumulation points (see [HLR11, Theorem 2.7]). We rather look at accumulation points of the directions of the roots. In order to do this, we will cut those directions by an hyperplane transverse to Φ^+ , i.e. an affine hyperplane that intersects all the directions of the roots. So we get points that are representatives of those directions, see for instance 3.

The discussion in §2 depends heavily on the basis \mathcal{B} . By [DHR13, Proposition 4.13], we can fix $\mathcal{B} = (e_1, \dots, e_{n+1})$ such that \mathcal{H} is transverse to Φ^+ . Moreover, \mathcal{B} and \mathcal{H} have the following properties:

⁵From Corollary 2.3.

⁶Otherwise we could restrict our study to the subspace $\text{span}(\Delta)$, in which Φ could be finite, affine or weakly hyperbolic, depending of the signature of the restriction of B to this subspace, see [DHR13] for more details.

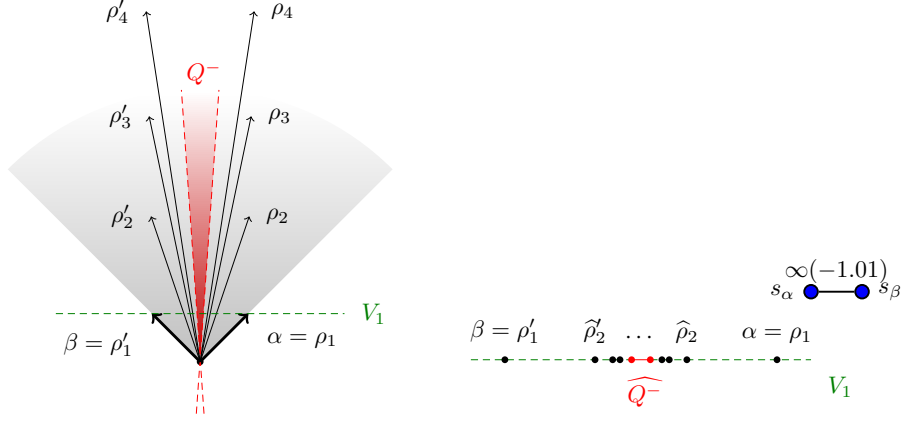


FIGURE 3. The isotropic cone Q and the first positive roots and normalized roots of an infinite based root system of rank 2 with $B(\alpha, \beta) = 1, 01$ in a Lorentzian 2-space.

- (1) $B(e_{n+1}, \alpha) < 0$ for all $\alpha \in \Delta$.
- (2) Denote by H the linear hyperplane directing \mathcal{H} , and for $v \in V \setminus H$, denote by \hat{v} the intersection point of the line $\mathbb{R}v$ with H (we also use the analog notation \hat{P} for a subset P of $V \setminus H$), see [HLR11, §2.1 and §5.2] for more details. With these notations we have:
 - (a) $\mathbb{H}_p^n = \widehat{Q^-} = Q^- \cap H$ and its boundary is $\partial\mathbb{H}_p^n = Q \cap H = \widehat{Q}$, by Proposition 2.2;
 - (b) for any $x \in \mathbb{H}_p^n$ and $w \in W$, $w \cdot x = \widehat{w(x)}$, by Corollary 2.3.

Our main objects of study are:

- The set of *normalized roots* $\widehat{\Phi} := \{\widehat{\beta} \mid \beta \in \Phi\}$, which is contained in the convex hull of the *normalized simple roots* $\widehat{\alpha}$ in $\widehat{\Delta}$, seen as points in V_1 . The normalized roots are representatives of the directions of the roots, or in other words, of the roots seen in the projective space $\mathbb{P}V$. In Figure 1, normalized roots are in blue, while the edges of the polytope $\text{conv}(\widehat{\Delta})$ are in green.
- The set $E(\Phi)$ of accumulation points of $\widehat{\Phi}$, to which tends the blue shape in Figure 1. For short, we call the elements of E , which are limit points of normalized roots, the *limit roots* of Φ .
- The action of W on $\text{conv}(E(\Phi)) \sqcup \widehat{\Phi}$ defined by $w \cdot x = \widehat{w(x)}$, see [DHR13, §2.3]. Since the root system is weakly hyperbolic W acts faithfully on $E(\Phi)$ by [DHR13, Theorem 6.1].

The set $E(\Phi)$ also enjoys some fractal properties as shown in [DHR13, §4], see also [HMN12]. It was those fractal properties that led us to exhibit the link with limit sets of discrete groups generated by hyperbolic reflections (Theorem 1.1).

3.4. Imaginary convex set and proof of Theorem 1.1. We prove here Theorem 1.1 from the introduction, whose precise statement is given below.

Theorem 3.3. *Let (Φ, Δ) be a weakly hyperbolic root system in a Lorentzian space (V, B) , and $W \subseteq \mathcal{I}(\mathbb{H}_p^n)$ its associated Coxeter group. Then the limit set $\Lambda(W)$ of W is equal to the set $E(\Phi)$ of limit roots of Φ .*

Since to study the limit set of W we may choose any point in \mathbb{H}_p^n , see §2.4, we will actually consider a particular subset that intersects \mathbb{H}_p^n : the *imaginary convex set*, see [DHR13, §2]. The imaginary convex set is the projective version of the imaginary cone that has been first introduced by Kac (see [Kac90, Ch. 5]) in the context of Weyl groups of Kac-Moody Lie algebras; this notion has been generalized afterwards to arbitrary Coxeter groups, first by Hée [Hée93], then by Dyer [Dye12] (see also Edgar’s thesis [Edg09] or Fu’s article [Fu11]). The definition we use here applies to any finitely generated Coxeter group, and is illustrated in Figure 4.

Definition 3.4. The *imaginary convex set* $Z(\Phi)$ is the W -orbit of the polytope

$$K := \{v \in \text{conv}(\widehat{\Delta}) \mid B(v, \alpha) \leq 0, \forall \alpha \in \Delta\}.$$

The imaginary convex set is intimately linked with the set of limit roots, see [DHR13, §2]:

$$\text{conv}(E(\Phi)) = \overline{Z} \subseteq \{v \in \mathcal{H} \mid B(v, v) \leq 0\} \subseteq \overline{Q^-}.$$

Moreover, since (Φ, Δ) is weakly hyperbolic, we know by [DHR13, Lemma 2.4] that the polytope K has non-empty interior. In particular K and Z intersect $\mathbb{H}_p^n = \overline{Q^-}$. Therefore, W acts on the non-empty set $Z \cap \mathbb{H}_p^n$ with the action of Corollary 2.3.

Now, the limit set of W is independent of the choice of the initial chosen point in \mathbb{H}_p^n , see §2.4. Thus, for any $v \in Z \cap \mathbb{H}_p^n$, the limit set of W :

$$\Lambda(W) = \text{Acc}(W \cdot v).$$

Still using the assumption that (Φ, Δ) is weakly hyperbolic, we have by [DHR13, Corollary 6.15(c)]:

$$(2) \quad E(\Phi) = \text{Acc}(W \cdot v), \quad \text{for } v \in Z$$

This proves Theorem 3.3 and therefore Theorem 1.1.

Remark 3.5. It is shown in [DHR13, Theorem 4.10], since the root system is weakly hyperbolic, that $E(\Phi)$ can be recovered easily from \overline{Z} : $E(\Phi) = \overline{Z} \cap Q$. This property is the key to prove Equation (2). Whether these two properties are true or not for root systems, when (V, B) is not a Lorentzian or Euclidean space, is an open question, see [DHR13, Question 4.9] and the proof of [DHR13, Corollary 6.15(c)].

3.5. Discrete groups generated by hyperbolic reflections and Coxeter groups. The word “hyperbolic” is often attached to the expression “Coxeter groups” in the literature. In the same vein, the relation between “discrete groups generated by hyperbolic reflections” and “hyperbolic Coxeter groups” is not always clearly transparent. It does not always seem to mean the same thing. For instance, in the article of Krammer [Kra09], “hyperbolic Coxeter group” means a Coxeter group attached to a weakly hyperbolic root system, whereas in Humphreys’ book [Hum90] this means a strict subclasses of Coxeter groups attached to a weakly hyperbolic root system. See also the difference in the use of these expressions between [Dav08, Rat06, AB08] or [Dol08]. We end this section by clarifying the relation between those terms.

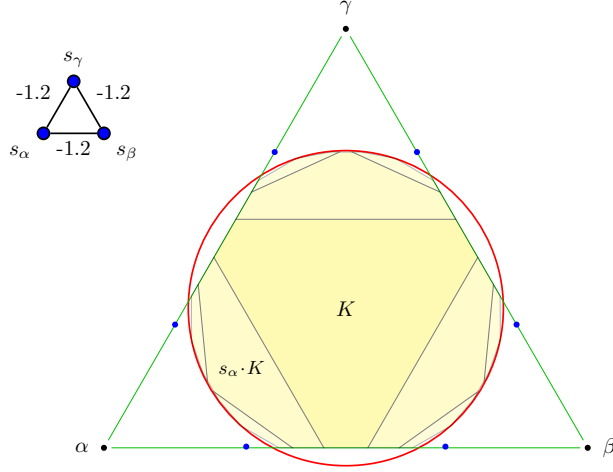


FIGURE 4. Picture in rank 3 of the first 9 normalized roots for the weakly hyperbolic root system with diagram in the upper left of the picture. In red is $\hat{Q} = \partial\mathbb{H}_p^2$ and in green is the boundary of $\text{conv}(\hat{\Delta})$. Some reflections hyperplanes for W , K and some of its images through reflections are also represented in shaded yellow.

3.5.1. Discrete reflection groups of hyperbolic isometries. A discrete reflection group on \mathbb{H}^n is a discrete subgroup of hyperbolic isometries $\Gamma \subseteq \mathcal{I}(\mathbb{H}^n)$ generated by (finitely many) hyperbolic reflections, see [VS93, Chapter 5, §1.2]. We explained before how a Coxeter group with a root system in a Lorentzian space has a representation as a discrete reflection group, see proposition 3.1. Conversely, we have the following theorem.

Theorem 3.6 (Vinberg [Vin71, VS93]). *Discrete reflection groups on \mathbb{H}^n are Coxeter groups that are associated to based root systems in Lorentzian spaces.*

This result due to Vinberg completes for spaces of constant curvature the classical result of Coxeter [Cox34] that shows that: (1) discrete reflection groups on the sphere (i.e. Euclidean vector space) are finite and are Coxeter groups; (2) discrete reflection groups on affine Euclidean space are Coxeter groups. Moreover, Coxeter classified those groups in the case of the sphere and of the affine Euclidean space, see [Hum90]. Such a classification is still unknown for Coxeter groups that arise as discrete reflection groups on \mathbb{H}^n . Only the subclass of *hyperbolic reflection groups* is classified, see below for more details.

Remark 3.7 (Remark on the proof). To show that a discrete reflection group Γ on \mathbb{H}^n is a Coxeter group is a bit more complicated than in the case of the sphere and of the affine Euclidean space, but the main steps are the same. To our knowledge, the theorem above is never stated precisely in these terms, but is clearly apparent in [VS93, Chapter 5, §1.1 and §1.2].

First, consider the hyperplane arrangement associated to Γ , i.e., the set of hyperplanes of the reflections in Γ . This hyperplane arrangement is locally finite and therefore decomposes \mathbb{H}^n into (hyperbolic) convex polyhedra (each of these is a

Dirichlet domain, see [VS93] or [Rat06, §6.6]). Pick one of these to be the *fundamental chamber*⁷ P , which is a convex polyhedron. Then this fundamental chamber is a *fundamental domain* for the action of Γ on \mathbb{H}^n and the angles between the facets that intersect⁸ are *submultiples of π* , see [VS93, Chapter 5, §1.1 and Proposition 1.4] or [Dol08, Theorem 2.1]. Then, by [Vin71, Theorem 2], we know that the exterior unitary (for B) normal vector in V associated to the facets of P form a simple system Δ and its orbit Φ is a root system. Finally, by denoting S the set of reflections across the facets of P , one deduces that (Γ, S) is a Coxeter system associated to the based root system (Φ, Δ) (see also [Kra09, Theorem 1.2.2]).

3.5.2. Fundamental polyhedron of discrete reflection groups of hyperbolic isometries. Let $W \subseteq \mathcal{I}(\mathbb{H}^n)$ be a discrete reflection group on \mathbb{H}^n , with associated root system (Φ, Δ) in (V, B) . A fundamental convex polyhedron P , i.e., a fundamental chamber as in the remark above, can be easily described with the help of the *Tits cone*. Since B is non-degenerate, we associate V and its dual. So

$$C := \{v \in V \mid B(v, \alpha) \geq 0, \forall \alpha \in \Delta\}$$

is a fundamental domain for the action of W on the *Tits cone* \mathcal{U} , which is the union of the W -orbit $W(C)$; we call C the *fundamental chamber*.

Remark 3.8. Observe that with Definition 3.4, $K = (-C) \cap \text{conv}(\hat{\Delta})$, and the imaginary convex set Z is contained in the negative of the Tits cone $-\mathcal{U}$, see Figure 5.

The following proposition is mostly a reinterpretation of classical results, see for instance [Kra09].

Proposition 3.9. *Let (Φ, Δ) be a root system in the Lorentzian $(n+1)$ -space (V, B) with associated Coxeter system (W, S) . Then*

- (i) $(-\mathcal{U}) \cap \mathbb{H}^n = \mathbb{H}^n$ and $(-\mathcal{U}) \cap \mathbb{H}_p^n = \mathbb{H}_p^n$.
- (ii) $P := (-C) \cap \mathbb{H}^n$ is a fundamental domain for the action of the discrete group of isometry W on \mathbb{H}^n .
- (iii) $(-C) \cap \mathbb{H}_p^n$ is a fundamental domain for the action of the discrete group of isometry W on \mathbb{H}_p^n described in Corollary 2.3.

Proof. By [Kra09, Proposition 4.6.1], we know that either $\mathcal{U} \cap Q^- = \{v \in Q^- \mid x_{n+1} < 0\}$ or $\mathcal{U} \cap Q^- = \{v \in Q^- \mid x_{n+1} > 0\}$. We have $B(e_{n+1}, \alpha) < 0$ for all $\alpha \in \Delta$, by item (1) of §3.3. Thus we have $e_{n+1} \in -C$. So $(-\mathcal{U}) \cap Q^- = \{v \in Q^- \mid x_{n+1} > 0\}$. Therefore $(-\mathcal{U}) \cap \mathbb{H}^n = \mathbb{H}^n$ and $(-\mathcal{U}) \cap \mathbb{H}_p^n = \mathbb{H}_p^n$. The rest of the proposition follows from the fact that $-C$ is a fundamental domain for $-\mathcal{U}$. \square

3.5.3. Hyperbolic Coxeter groups. Let W be a discrete reflection group on \mathbb{H}^n . If the fundamental chamber P , see Proposition 3.9, is of finite volume, then W is called a *hyperbolic Coxeter group*; if P is moreover compact, then W is called a *compact hyperbolic Coxeter group*, see [Rat06, Hum90, Dav08]. Their associated root systems are respectively called *hyperbolic root systems* and *compact hyperbolic root systems*. These two classes of root systems can be characterized as strict subclasses of weakly hyperbolic root systems, see for instance [Dye12, §9] or [DHR13, §4.1]. Let (Φ, Δ) be a weakly hyperbolic root system, then

⁷Also called *fundamental convex polyhedron in the literature*, see [Rat06, VS93]

⁸See Remark 3.2.

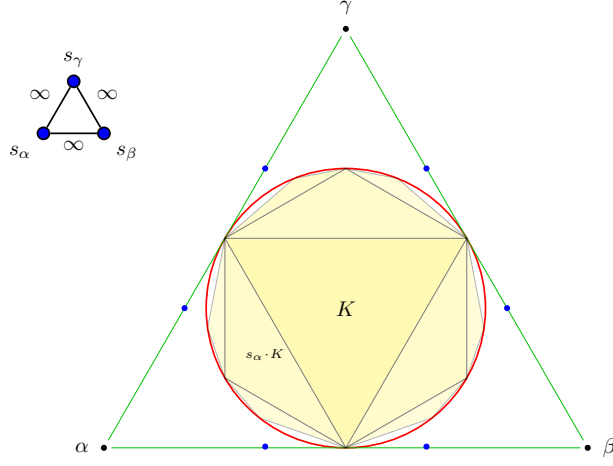


FIGURE 5. A *hyperbolic root system*. Picture in rank 3 of the first normalized root converging toward the set of limits of roots $E(\Phi)$ for the root system with diagram in the upper left of the picture. In red is $\hat{Q} = \partial\mathbb{H}_p^2$. Reflections hyperplanes for W and K are also represented. Note that here $K \cap \mathbb{H}_p^2 = -C \cap \mathbb{H}_p^2$ is not compact but has a finite volume in \mathbb{H}_p^2 , see §3.5.3.

- (Φ, δ) is *hyperbolic* if and only if B restricts to a positive form on each proper faces of $\text{cone}(\Delta)$, i.e. $-C \subseteq \text{conv}(\hat{\Delta})$; an example is given in Figure 5
- (Φ, Δ) is *compact hyperbolic* if and only if B restricts to a positive definite form on each proper faces of $\text{cone}(\Delta)$, i.e. $-C \subseteq \text{int}(\text{conv}(\hat{\Delta}))$; an example is given in Figure 6. In particular, a compact hyperbolic Coxeter group cannot have a relation of the form $m_{s,t} = \infty$.

The example in Figure 4 is neither compact hyperbolic nor hyperbolic, and $K \cap \mathbb{H}_p^2 \subsetneq -C \cap \mathbb{H}_p^2$.

Remark 3.10. We should be careful with the terminology here, because the properties studied often depend on the geometry, i.e., on W as a discrete reflection group on \mathbb{H}^n and not only as an abstract Coxeter group.

- (1) For this reason, we do not say that a Coxeter group associated to a weakly hyperbolic root system is a “weakly hyperbolic Coxeter group”: for example, the universal Coxeter group of rank 4 admits a representation with a weakly hyperbolic root system, but it also admits a representation with a root system of signature $(2, 2)$, which is therefore not weakly hyperbolic, see [DHR13, Remark 4.3 and Figure 5].
- (2) In the case of rank 3 Coxeter groups, we should avoid using the terminology “hyperbolic Coxeter groups”. Indeed, for instance, the universal Coxeter group of rank 3 admits a geometric representation attached to a weakly hyperbolic, non hyperbolic, root system (see Figure 4), and a geometric representation attached to a hyperbolic root system (see Figure 5). This can happen if and only if at least one of the labels of the Coxeter graph is ∞ , i.e., there are several choices for the value of $B(\alpha, \beta) \leq -1$ for some simple roots $\alpha, \beta \in \Delta$.

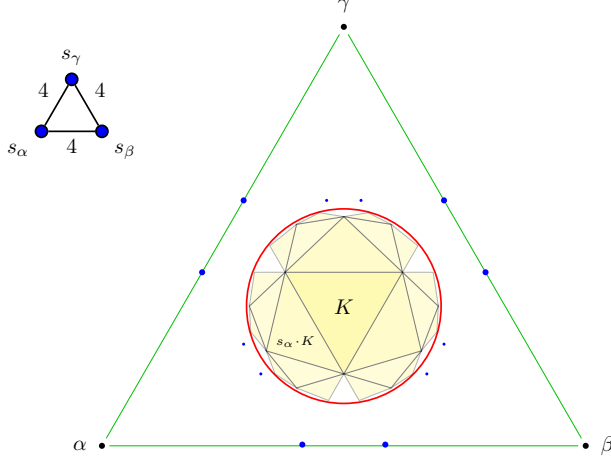


FIGURE 6. A *compact hyperbolic root system*. Picture in rank 3 of the first normalized root converging toward the set of limits of roots $E(\Phi)$ for the root system with diagram in the upper left of the picture. In red is $\hat{Q} = \partial\mathbb{H}_n^p$. Reflections hyperplanes for W and K are also represented. Note that here $K \cap \mathbb{H}_p^2 = -C \cap \mathbb{H}_p^2$ is compact in \mathbb{H}_p^2 .

- (3) The two situations above cannot happen with hyperbolic Coxeter groups of rank ≥ 4 , or with compact hyperbolic Coxeter groups, because the hyperbolic hyperplanes associated to simple roots cannot be parallel or ultra-parallel in those cases (they always intersect). Indeed, these types of Coxeter groups are classified, see [Hum90, §6.9], and none of them admits a label ∞ in the Coxeter graph.

Hyperbolic Coxeter groups, and compact hyperbolic Coxeter groups, have been classified⁹, see [Hum90, §6.9]. From this classification and [DHR13, Theorem 4.4] we easily deduce the following proposition.

Proposition 3.11. *Let (Φ, Δ) be an irreducible indefinite¹⁰ root system of rank ≥ 4 , with Coxeter system (W, S) . Then W is hyperbolic if and only if $E(\Phi) = \hat{Q}$.*

In other words, with Theorem 1.1 we obtain the following corollary.

Corollary 3.12. *With the same notation as the proposition above, we have W is hyperbolic if and only if its limit set $\Lambda(W) = \hat{Q}$.*

4. AN EXAMPLE: UNIVERSAL COXETER GROUP AND APOLLONIAN GASKETS

As an illustration, we describe here, in the light of the preceding sections, the limit set appearing in Figure 1, which is an Apollonian gasket.

⁹A far as we know, Coxeter groups associated to weakly hyperbolic root systems have not been classified.

¹⁰This means that the Coxeter graph is connected and that the Coxeter group is neither finite nor affine, see for instance [DHR13] for more details.

4.1. Conformal models of the hyperbolic space. We now introduce two conformal models of the hyperbolic space, the *conformal ball model* and the *upper halfspace model* that turn out to be more practical to deal with the geometry because their isometries are Möbius transformations. For details we refer the reader to [Rat06, Chapter 4] and [BP92, Chapter A]. We use the notation $\|\cdot\|$ for the Euclidean norm of \mathbb{R}^n .

4.1.1. Inversions and the Möbius group. In the Euclidean space \mathbb{R}^n endowed with its standard scalar product, let $\mathcal{S}(a, r)$ denotes a sphere with center a and radius r . The *inversion* with respect to $\mathcal{S}(a, r)$ is the map:

$$\begin{aligned} i_{a,r} : \mathbb{R}^n \setminus \{a\} &\longrightarrow \mathbb{R}^n \setminus \{a\} \\ x &\longmapsto a + r^2 \cdot \frac{x-a}{\|x-a\|^2} \end{aligned}$$

It is an involutive diffeomorphism that is conformal and changes spheres into spheres. It extends to an involution $I_{a,r}$ of the one point compactification $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ by setting $I_{a,r}(a) = \infty$ and $I_{a,r}(\infty) = a$, which is a conformal involutive diffeomorphism once $\overline{\mathbb{R}^n}$ is given its standard diffeomorphic and conformal structures. Then $I_{a,r}$ changes spheres/hyperplanes into spheres/hyperplanes. Its set of fixed points is the whole sphere $\mathcal{S}(a, r)$, and conformality implies that a sphere/hyperplane H is stable under $I_{a,r}$ if and only if H intersects $\mathcal{S}(a, r)$ orthogonally. Whenever $I_{a,r}$ changes the sphere $\mathcal{S}_{b,\rho}$ into an hyperplane H then $I_{a,r} \circ I_{b,\rho} \circ I_{a,r}^{-1}$ is the Euclidean (orthogonal) reflection with respect to H . The *Möbius group* of $\overline{\mathbb{R}^n}$ (or \mathbb{R}^n) is defined as the group generated by all inversions and reflections in $\overline{\mathbb{R}^n}$.

4.1.2. The Conformal Ball Model \mathbb{H}_c^n . Consider the open unit ball embedded in the hyperplane $\mathbb{R}^n \times \{0\}$ of V :

$$D^n = \{(x_1, \dots, x_{n+1}) \in V \mid x_{n+1} = 0 \text{ and } x_1^2 + \dots + x_n^2 = 1\}$$

and the *stereographic projection* c with respect to $-e_{n+1}$ of $\mathbb{R}^n \times \mathbb{R}_+^*$ onto $\mathbb{R}^n \times \{0\}$:

$$\begin{aligned} c : \mathbb{R}^n \times \mathbb{R}_+^* &\longrightarrow \mathbb{R}^n \times \{0\} \\ (x_1, \dots, x_{n+1}) &\longmapsto \frac{(x_1, \dots, x_n, 0)}{1+x_{n+1}} \end{aligned}$$

One verifies that c restricted to the hyperboloid model \mathbb{H}^n is a diffeomorphism onto D^n (cf. [Rat06, BP92]). Once D^n is endowed with the pull-back metric (the Riemannian metric $ds = \frac{dx}{1-\|x\|^2}$) one obtains the *conformal ball model* of the hyperbolic space, that we denote by \mathbb{H}_c^n (cf. Figure 7).

The (hyperbolic) hyperplanes in \mathbb{H}_c^n are the intersections with D^n of the Euclidean spheres and hyperplanes in $\mathbb{R}^n \times \{0\}$ that are perpendicular to the boundary sphere $\partial\mathbb{H}_c^n := \partial D^n$. The hyperbolic reflection with axis the hyperplane H is the restriction of the inversion with respect to the Euclidean sphere or hyperplane in $\mathbb{R}^n \times \{0\}$ containing H . It turns out that the group of isometries $\mathcal{I}(\mathbb{H}_c^n)$ is the subgroup of the Möbius group of $\mathbb{R}^n \times \{0\}$ that leaves invariant D^n or, equivalently, generated by inversions/reflections with respect to hyperplanes/spheres that are perpendicular to the boundary. The model is conformal: the hyperbolic and Euclidean angles are the same.

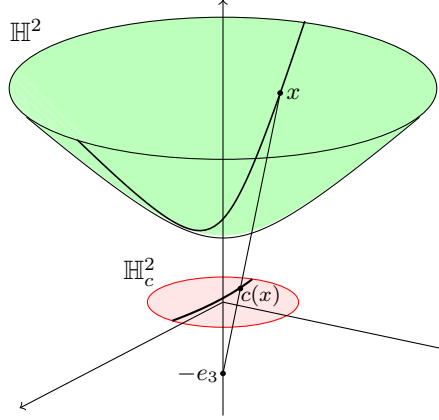


FIGURE 7. The conformal disk model.

The map $c \circ p : \mathbb{H}_p^n \rightarrow \mathbb{H}_c^n$ is an isometry from the projective to the conformal ball models and a simple computation shows that:

$$c \circ p(x_1, \dots, x_n, 1) = \frac{1 - \sqrt{1 - x_1^2 - \dots - x_n^2}}{x_1^2 + \dots + x_n^2} \cdot (x_1, \dots, x_n, 0)$$

so that it obviously extends to an homeomorphism from $\overline{\mathbb{H}_p^n} = \mathbb{H}_p^n \cup \partial\mathbb{H}_p^n$ to $\overline{\mathbb{H}_c^n} = \mathbb{H}_c^n \cup \partial\mathbb{H}_c^n$ that restricted to $\partial\mathbb{H}_p^n \rightarrow \partial\mathbb{H}_c^n$ is the translation with vector $-e_{n+1}$.

4.1.3. *The Upper Half Space Model \mathbb{H}_u^n .* Consider the differentiable map:

$$\begin{aligned} D^n &\longrightarrow \mathbb{R}^n \\ u : x &\longmapsto 2 \frac{x + e_n}{\|x + e_n\|^2} - e_n \end{aligned}$$

One verifies (cf. [BP92], Chapter A) that u is a diffeomorphism from D^n onto the open upper half-space: $\mathbb{R}^{n-1} \times \mathbb{R}_+^* = \{x \in \mathbb{R}^n \mid x_n > 0\}$, which, in fact, is the inversion with respect to the sphere with radius $\sqrt{2}$ and center $-e_n$ (cf. §4.1.1 and Figure 8). Once D^n is identified with the conformal ball model \mathbb{H}_c^n and $\mathbb{R}^{n-1} \times \mathbb{R}_+^*$ is endowed with the pull-back metric with respect to u^{-1} , we obtain the *upper half-space model* \mathbb{H}_u^n of the hyperbolic space with Riemannian metric $ds^2 = dx_1^2 + \dots + dx_{n-1}^2 + dx_n^2/x_n^2$. The hyperplanes in \mathbb{H}_u^n are euclidean half-spheres with centers on the boundary $\mathbb{R}^{n-1} \times \{0\}$ as well as vertical affine hyperplanes. The model is conformal: hyperbolic angles agree with Euclidean ones. A reflection with respect to a hyperplane H is a Euclidean reflection with respect to H (when H is a 'vertical' Euclidean hyperplane) or an inversion with respect to H (when H is a 'half-sphere').

The group of isometries of \mathbb{H}_u^n is the subgroup of the Möbius group of \mathbb{R}^n that stabilizes $\mathbb{R}^{n-1} \times \mathbb{R}_+^*$, or equivalently, that one generated by inversions/reflections with respect to spheres/hyperplanes perpendicular to the boundary $\mathbb{R}^{n-1} \times \{0\}$.

The hyperbolic boundary $u(\partial\mathbb{H}_c^n)$ of \mathbb{H}_u^n is the one point compactification $\partial\mathbb{H}_u^n := (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$.

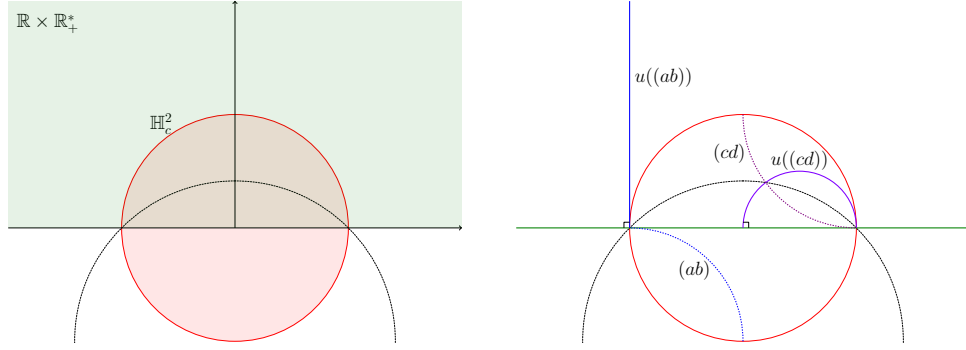


FIGURE 8. The inversion u with respect to the sphere with center $-e_n$ and radius $\sqrt{2}$ (in dash) sends the conformal disk model (in red) onto the upper half-plane model (in green). On the right side two infinite geodesics in \mathbb{H}_c^2 , (ab) and (cd) , and their images by u , which are geodesics of \mathbb{H}_u^2 .

4.2. Representation of the universal Coxeter group of rank 3 as a discrete subgroup of isometries of \mathbb{H}^2 with finite covolume.

Consider an ideal triangle (abc) in \mathbb{H}_c^2 , with a, b, c three distinct points in $\partial\mathbb{H}_c^2$. Its sides (ab) , (ac) and (bc) are infinite geodesics that are pairwise parallel, therefore with angles 0, and (abc) has a finite area equal to π ([Rat06], §3.5, Lemma 4). Let $G \subset \mathcal{I}(\mathbb{H}_c^2)$ be the hyperbolic reflection group generated by the (hyperbolic) reflections $s_{(ab)}$, $s_{(ac)}$, $s_{(bc)}$ with respect to the sides of (abc) . The group G is a generalized simplex reflection group in the sense of [Rat06] (cf. §7.3). The interior \mathcal{P} of the domain delimited by (abc) is a fundamental region (cf. [Rat06], §6) for the action of G on \mathbb{H}_c^2 so that G is a discrete group (Theorem 6.6.3, [Rat06]), and G together with its generating set is the universal Coxeter group of rang 3:

$$< s_{(ab)}, s_{(ac)}, s_{(bc)} \mid s_{(ab)}^2 = s_{(ac)}^2 = s_{(bc)}^2 = 1 >$$

isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ (Theorem 7.1.4, [Rat06]); see also Figure 5.

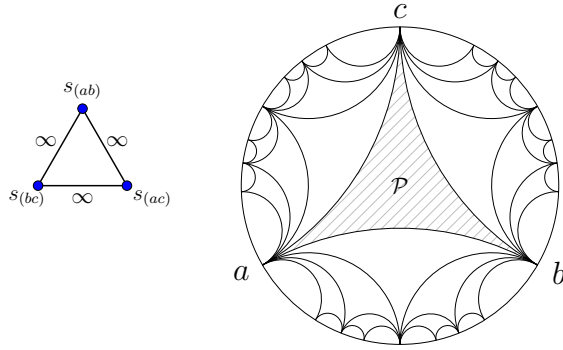


FIGURE 9. The fundamental domain \mathcal{P} and some of its G -translated for the action of G onto the conformal disk model \mathbb{H}_c^2 .

The limit set $\Lambda(G) = \overline{G \cdot x} \setminus G \cdot x$ of G lies in $\partial\mathbb{H}_c^2$ (Theorem 12.1.2, [Rat06]); it turns out that $\Lambda(G)$ is the whole of $\partial\mathbb{H}_c^2$:

Lemma 4.1. *The limit set of G is $\Lambda(G) = \partial\mathbb{H}_c^2$.*

The fact that the limit set of G is the whole boundary $\partial\mathbb{H}_c^2$ also follows from the fact that this geometrical representation of the universal Coxeter group G of rank 3 is a hyperbolic Coxeter group, see Remark 3.10 in §3.5.3. This also illustrates a more general result stated in [DHR13, Theorem 4.4]. We give here a direct proof for convenience.

Proof. Let $\omega \in \partial\mathbb{H}_c^2$; it suffices to prove that there exists $x \in \mathbb{H}_c^2$ and a sequence $(u_n)_n$ in $G \cdot x$ that converges to ω . Let x be an arbitrary point in the fundamental domain \mathcal{P} . We suppose without loss of generality that $\omega \neq a$ and consider the upper half-space model \mathbb{H}_u^2 with a sent to ∞ . In this model the geodesics (ab) and (ac) become vertical lines that enclose a closed region \mathcal{W}^0 in $\mathbb{R} \times \mathbb{R}_+$, and $\mathbb{R} \times \mathbb{R}_+$ is the union of the countable family $(\mathcal{W}^n)_{n \in \mathbb{Z}}$ where $\mathcal{W}^n = (s_{(ac)} \circ s_{(ab)})^n(\mathcal{W}^0)$ are G -translated of \mathcal{W}^0 .

Let $\mathcal{D}^0 = \mathcal{W}^0 \setminus \mathcal{P} \subset \mathcal{W}^0$; \mathcal{D}^0 is the half-disk domain in $\mathbb{R} \times \mathbb{R}^+$ delimited by (bc) and it contains an uppermost G -translated of \mathcal{P} : the "triangular" domain $\mathcal{P}^0 = s_{(bc)}(\mathcal{P})$ with side (bc) (hatched in Figure 10). Let x^0 denotes the unique point in $\mathcal{P}^0 \cap G \cdot x$. The two other sides of \mathcal{P}^0 , namely the infinite parallel hyperbolic geodesics $s_{(bc)}(ac)$ and $s_{(bc)}(ab)$ cut \mathcal{D}^0 into \mathcal{P}^0 and two half-disk domains \mathcal{D}_0^0 and \mathcal{D}_1^0 . As above \mathcal{D}_i^0 ($i = 0, 1$) decomposes into a triangular domain \mathcal{P}_i^0 , which is a G -translated of \mathcal{P} , and two half-disk domains \mathcal{D}_{i0}^0 and \mathcal{D}_{i1}^0 and we set $x_i^0 = \mathcal{P}_i^0 \cap G \cdot x$, and so on: in this construction for each finite sequence σ of 0 and 1, \mathcal{D}_σ^0 is a half-disk domain in \mathcal{D}^0 that decomposes into a G -translated of \mathcal{P} and two half-disk domains $\mathcal{D}_{\sigma 0}^0$ (the left one) and $\mathcal{D}_{\sigma 1}^0$ (the right one) and we set x_σ^0 the unique point in $\mathcal{P}_\sigma^0 \cap G \cdot x$. A similar construction yields for all $n \in \mathbb{Z}$ and for all finite sequence σ in $\{0, 1\}$, a half-disk domain $\mathcal{D}_\sigma^n = (s_{(ac)} \circ s_{(ab)})^n(\mathcal{D}_\sigma^0)$ and a point $x_\sigma^n = (s_{(ac)} \circ s_{(ab)})^n(x_\sigma^0)$ that lies in $\mathcal{D}_\sigma^n \cap G \cdot x$ (see Figure 10).

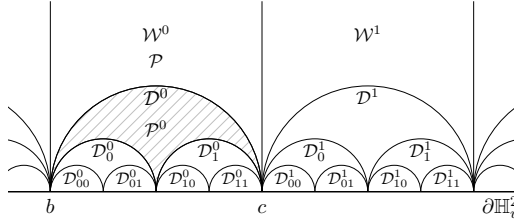


FIGURE 10

We now define by induction a sequence $(u_n)_n$ in $G \cdot x$ that converges to ω . By construction ω lies in (at least) one domain \mathcal{D}^k , say \mathcal{D}^0 , then set $u_0 = x^0$. Whenever σ is a (possibly empty) finite sequence of length $n \in \mathbb{N}$ in $\{0, 1\}$ such that $\omega \in \mathcal{D}_\sigma^0$ and $u_n = x_\sigma^0$ then necessarily $\omega \in \mathcal{D}_{\sigma 0}^0$ or $\omega \in \mathcal{D}_{\sigma 1}^0$ (possibly both), say $\mathcal{D}_{\sigma 1}^0$, then set $u_{n+1} = x_{\sigma 1}^0$. By construction $(u_n)_n$ lies in $G \cdot x$ and $\forall n \in \mathbb{N}$, $\|\omega - u_n\| < \|b - c\|/2^n$ (for $\|\cdot\|$ the Euclidean norm) therefore $\lim u_n = \omega$. That proves that $\omega \in \Lambda(G)$. \square

4.3. The Apollonian gasket. We consider in the conformal disk model \mathbb{H}_c^2 the three infinite geodesics in (ab) , (ac) and (bc) , as above (see Figure 9).

Lemma 4.2. *For all three distinct points a, b, c in $\partial\mathbb{H}_c^2$:*

- (i) *there exists unique horocycles h_a, h_b, h_c with limit points a, b, c that are pairwise tangent.*
- (ii) *(ab) intersects h_a and h_b perpendicular onto the point $h_a \cap h_b$.*
- (iii) *There exists a unique circle \mathcal{C} passing through the intersection points $h_a \cap h_b, h_a \cap h_c$ and $h_b \cap h_c$; moreover \mathcal{C} is tangent to the three geodesics $(ab), (ac)$ and (bc) .*

Proof. In the upper half space model \mathbb{H}_u^2 with c sent to ∞ (see Figure 11) the horocycle h_c becomes the horizontal line $y = 2r$, h_a and h_b become circles tangent to the boundary $\mathbb{R} \times \{0\}$ respectively in a and b ; therefore the horocycles are pairwise tangent if and only if h_a, h_b both have radius r and $2r = \|a - b\|$. This proves (i). The geodesics (ac) and (bc) become vertical lines $x = a$ and $x = b$ while (ab) is a

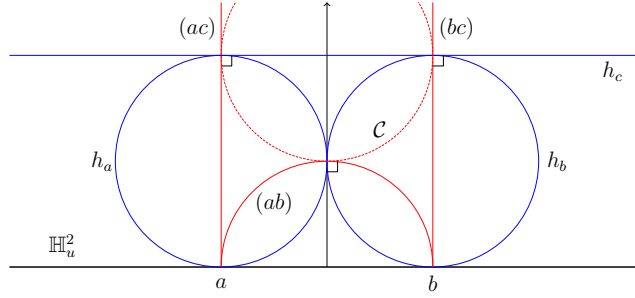
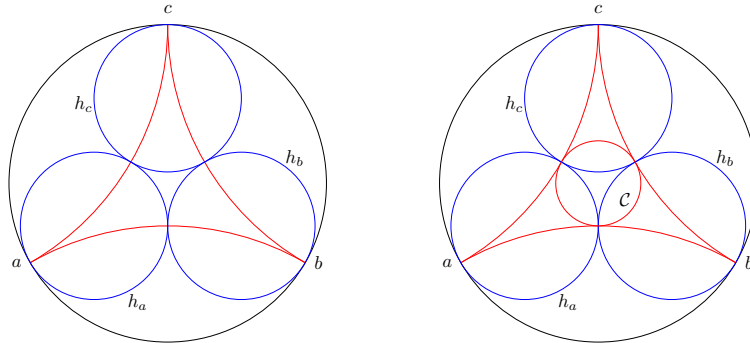


FIGURE 11

half-circle with diameter the segment $[a, b]$. One obtains (ii).

The three tangency points are not aligned, hence there exists a unique circle \mathcal{C} passing through them; it has diameter the segment with extremities $h_a \cap h_c$ and $h_b \cap h_c$, radius r , and is perpendicular to the three horocycles; with (ii), \mathcal{C} is tangent to the three geodesics. This proves (iii). \square

FIGURE 12. The geodesics $(ab), (ac), (bc)$, horocycles h_a, h_b, h_c and the circle \mathcal{C} in the conformal disk model.

In the conformal disk model \mathbb{H}_c^2 the three geodesics (ab) , (ac) and (bc) lie into three unique circles, respectively $\mathcal{C}_{(ab)}$, $\mathcal{C}_{(ac)}$ and $\mathcal{C}_{(bc)}$ of the Euclidean plane. Let \overline{G} be the subgroup of the Möbius group of \mathbb{R}^2 generated by the inversions with respect to $\mathcal{C}_{(ab)}$, $\mathcal{C}_{(ac)}$, $\mathcal{C}_{(bc)}$, and the circle \mathcal{C} given by Lemma 11. The group G of § 4.2 identifies with the subgroup generated by the inversions across $\mathcal{C}_{(ab)}$, $\mathcal{C}_{(ac)}$, and $\mathcal{C}_{(bc)}$ and accordingly \overline{G} is generated by $s_{(ab)}$, $s_{(ac)}$, $s_{(bc)}$ and the inversion $s_{\mathcal{C}}$ with respect to \mathcal{C} .

Since the inversion with respect to \mathcal{C} preserves a circle/line if and only if \mathcal{C} intersects the circle/line with right angles (cf. §4.1.1), Lemma 4.2 implies that $s_{\mathcal{C}}$ preserves each of the horocycles h_a , h_b , h_c ; each one of the hyperbolic reflections $s_{(ab)}$, $s_{(ac)}$ and $s_{(bc)}$ preserves the two horocycles that intersect their axis (respectively h_a , h_b , h_a , h_c and h_b , h_c) and moves the remaining one (respectively h_c , h_b and h_a) to an horocycle that remains tangent to the two others. The orbit of the three horocycles h_a , h_b , h_c and of $\partial\mathbb{H}_c^2$ under the action of \overline{G} yields a configuration of pairwise tangent or disjoint circles in $\overline{\mathbb{H}}_c^2$, see Figure 13. This configuration is called an *Apollonian gasket* \mathcal{A} and is widely studied in the literature, see for instance [Hir67, Max82, Gr et al.05, KH11, Kir13].

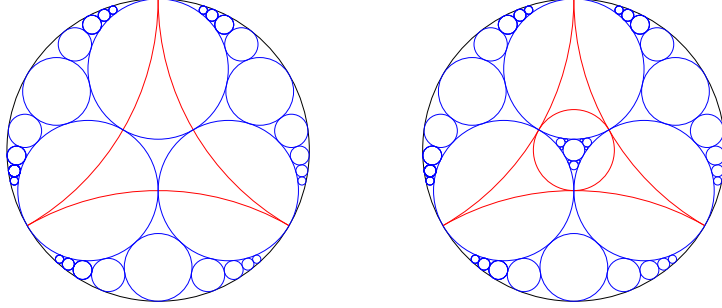


FIGURE 13. On the left (respectively right) some of the orbit of the three horocycles (respectively and of the boundary circle) under the action of G (respectively \overline{G}). The complete orbit on the right figure yields the *Apollonian gasket*.

4.4. Discrete representation in $\mathcal{I}(\mathbb{H}^3)$ of the universal Coxeter group with rank 4. Consider the universal Coxeter group of rank 4 and its representation as a discrete subgroup of $O_B(V)$ with $(V, B) = \mathbb{R}^{3,1}$ and simple system $\Delta = \{\alpha, \beta, \gamma, \delta\}$ such that for all distinct $\chi, \xi \in \Delta$, $B(\chi, \xi) = -1$. In Figures 1 and 14 are represented the polytope $\text{conv}(\widehat{\Delta})$ and the unit ball $D_1^3 = \widehat{Q}$ that we identify here with \mathbb{H}_p^3 .

As discussed in § 3 that makes the Coxeter group Γ act on the projective ball model by isometry (the action is given in Corollary 2.3), yielding a discrete faithful representation of Γ in $\mathcal{I}(\mathbb{H}_p^3)$.

A direct computation shows that the reflection s_α acts as the hyperbolic reflection across the hyperbolic plane $H_\alpha = \alpha^\perp \cap \mathbb{H}_p^3$ passing through the middles of the three edges issued from the vertex α of the tetrahedron; indeed, for example, $B(\alpha, \frac{\alpha+\beta}{2}) = \frac{1}{2}(B(\alpha, \alpha) + B(\alpha, \beta)) = \frac{1}{2}(1 - 1) = 0$ and the same computation shows that the middles of the three edges issued from α lie in α^\perp . Consider also the reflection planes H_β , H_γ , H_δ (in red in Figure 14) respectively associated to s_β , s_γ and s_δ ,

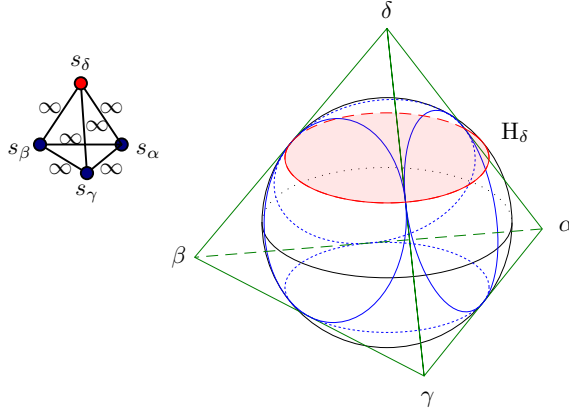


FIGURE 14. The unit ball D_1^3 with interior identified with the projective disk model \mathbb{H}_p^3 ; $\text{conv}(\hat{\Delta})$ is a regular tetrahedron such that the unit sphere ∂D_1^3 passes through the middles of its edges. In blue the circles on which ∂D_1^3 intersects $\text{conv}(\hat{\Delta})$. In red the plane H_δ in \mathbb{H}_p^3 passing through 3 of these points.

passing through the middles of the edges adjacent respectively to the vertices β , γ and δ . They are pairwise parallel and non ultra-parallel (they meet on the boundary $\partial\mathbb{H}_p^3$), which can also be seen by $\cosh d(H_\alpha, H_\beta) = |B(\alpha, \beta)| = 1$.

The boundary sphere $\partial\mathbb{H}_p^3$ intersects the faces of the tetrahedron onto circles; for $\chi = \alpha, \beta, \gamma, \delta$ let us denote by h_χ the intersection circle on the face of the tetrahedron opposite to the vertex χ (in blue in Figure 14).

In the upper half-space model, the planes H_α , H_β , H_γ and H_δ yield a configuration of 4 half-spheres that are pairwise tangent, see Figure 15. The action of Γ restricts on $\partial\mathbb{H}_u^3$ as the action of the subgroup \overline{G} of the Möbius group on the plane (see §4.3).

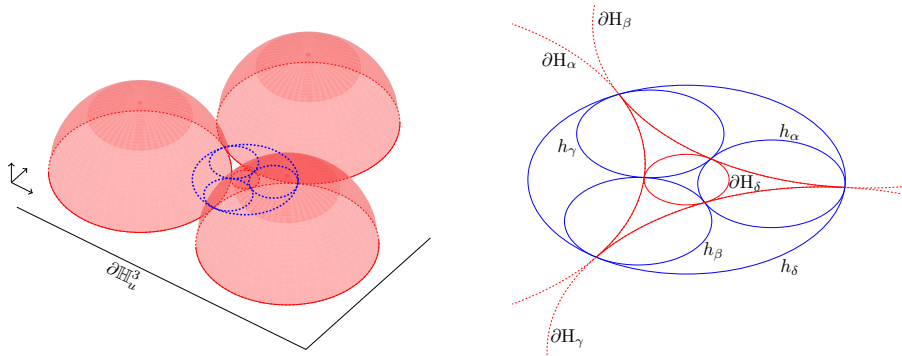


FIGURE 15. On the left picture: in red the four hyperbolic planes H_α , H_β , H_γ and H_δ (the small one in between) in the upper half-space model \mathbb{H}_u^3 , and in blue the four circles h_α , h_β , h_γ and h_δ in $\partial\mathbb{H}_u^3$. On the right picture their intersections with $\partial\mathbb{H}_u^3$.

Proposition 4.3. *The limit set $\Lambda(\Gamma)$ of Γ in $\overline{\mathbb{H}_u^3}$ is the closure in $\partial\mathbb{H}_u^3$ of the Apollonian gasket \mathcal{A} .*

Proof. The two hyperplanes H_α and H_δ are parallel so denote by $x_0 \in \partial\mathbb{H}_u^3$ their asymptotic point. Hence the composite of the two reflections with respect to H_α and H_δ is a transformation of parabolic type with limit point $x_0 \in \partial\mathbb{H}_u^3$ (cf. Propositions A.5.12 and A.5.14 of [BP92]). According to Theorem 12.1.1 of [Rat06], $x_0 \in \Lambda(\Gamma)$. Note that x_0 lies in the interior disk of $\mathbb{R}^2 \approx \partial\mathbb{H}_u^3 \setminus \{\infty\}$ delimited by the circle h_δ .

The action of Γ on \mathbb{H}_u^3 naturally extends to a conformal action on $\overline{\mathbb{H}_u^3}$ that is the Poincaré extension of the action of the subgroup \overline{G} of the Möbius group on $\partial\mathbb{H}_u^3 \approx \mathbb{R}^2$ (cf. §4.1.3 as well as Theorem 4.4.1 of [Rat06]). As in §4.3 denote by G the subgroup of \overline{G} generated by the three inversions with respect to the spheres ∂H_α , ∂H_β and ∂H_γ ; G acts on the interior of the disk delimited by ∂H_δ as the group G of §4.2 acts on \mathbb{H}_c^2 . In particular $\Lambda(\Gamma)$ contains $\overline{G \cdot x_0} \setminus G \cdot x_0$ that equals h_δ (Proposition 4.1).

After conjugating Γ by the reflection with respect to H_α (respectively H_β , H_γ) the same argument applies to show that $\Lambda(\Gamma)$ contains also h_α (respectively h_β , h_γ). Hence the Apollonian gasket \mathcal{A} , as seen in §4.3, which is the orbit of $h_\alpha \cup h_\beta \cup h_\gamma \cup h_\delta$ under the action of \overline{G} , is a Γ -invariant subset of $\Lambda(\Gamma)$; since $\Lambda(\Gamma)$ is closed in $\partial\mathbb{H}_u^3$ (Theorem 12.1.2, Corollary 1 of [Rat06]) the closure $\overline{\mathcal{A}}$ of \mathcal{A} in $\partial\mathbb{H}_u^3$ is a closed Γ -invariant subset of $\partial\mathbb{H}_u^3$ contained in $\Lambda(\Gamma)$. Since $\Lambda(\Gamma)$ is infinite, Γ is non elementary (cf. Theorem 12.2.1 of [Rat06]) and therefore any closed Γ -invariant subset of $\partial\mathbb{H}_u^3$ contains $\Lambda(\Gamma)$ (Theorem 12.1.3 of [Rat06]). Hence $\Lambda(\Gamma)$ equals $\overline{\mathcal{A}}$. \square

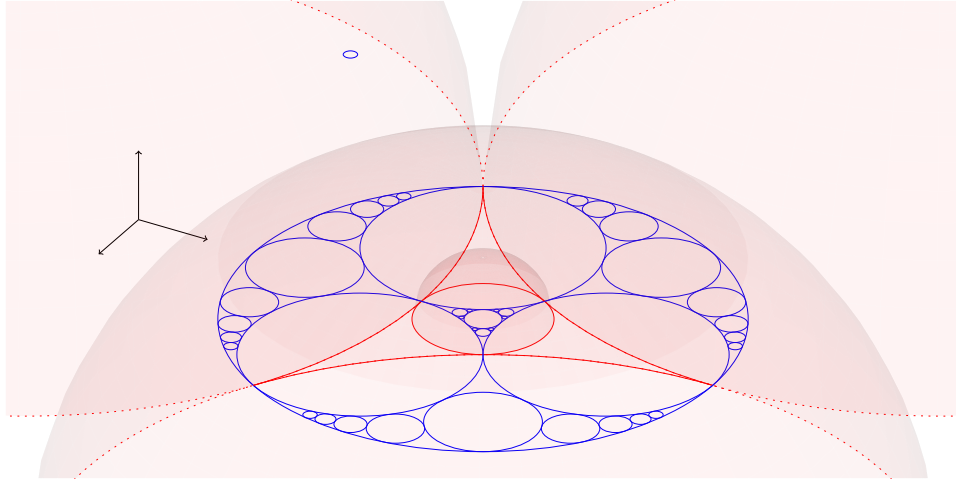


FIGURE 16. The Apollonian gasket in $\partial\mathbb{H}_u^3$ whose closure is the limit set of Γ . In both the conformal and projective ball models one obtains as limit set the Apollonian packing of the sphere as in Figure 1.

The closure $\overline{\mathcal{A}}$ of \mathcal{A} is the complement in the closed external disk (delimited by h_δ) of the union of the interiors of disks delimited by the circles of the gasket.

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